PERGAMON

International Journal of Solids and Structures 36 (1999) 1719–1734

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

An energy-based damage model of geomaterials—I. Formulation and numerical results

G. Swoboda[a,*], Q. Yang[b]

[a]*University of Innsbruck, Innsbruck, Austria*
[b]*Tsinghua University, Beijing, China*

Received 30 November 1996; in revised form 2 October 1997

Abstract

In this paper, an anisotropic damage model is established in strain space to describe the behaviour of geomaterials under compression-dominated stress fields. The research work focuses on rate-independent and small-deformation behaviour during isothermal processes. It is emphasized that the damage variables should be defined microstructurally rather than phenomenologically for geomaterials, and a second-order "fabric tensor" is chosen as the damage variable. Starting from it, a one-parameter damage-dependent elasticity tensor is deduced based on tensorial algebra and thermodynamic requirements; a fourth-order damage characteristic tensor, which determines anisotropic damaging, is deduced within the framework of Rice's (1971) "normality structure" in Part II of this paper. An equivalent state is developed to exclude the macroscopic stress/strain explicitly from the relevant constitutive equations. Finally, some numerical results are worked out to illustrate the mechanical behaviour of this model. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

Geomaterials (rock, concrete, ...) have complex mechanical behaviour, such as stress-induced anisotropy, hysteresis, dilatancy, irreversible and strongly path-dependent stress–strain relations, which is generally associated with the existence of a great deal of micro- and macro-cracks and their propagation. Continuum damage mechanics (CDM), which employs some continuum variables to describe the micro-defects, has been an appealing framework for modeling geomaterials, see e.g. Ju (1989), Dragon and Mroz (1979), Dragon et al. (1993), Kawamoto et al. (1988), Zhang (1992) and Stumvoll and Swoboda (1993).

In most of these models, the damage variables are defined through a phenomenological way

* University Innsbruck, Technikerstrasse 13, A-6020 Innsbruck, Austria. Fax: 0043 5125072999. E-mail: swoboda@cgie.uibk.ac.at.

PII: S0020-7683(98)00036-5

originating from the classical Kachanov's (1958) damage variable. Although they are often interpreted as the net area reduction caused by the distributed microscopic cracks and cavities and associated with a net stress (see e.g. Lemaitre, 1990), these damage variables are essentially measures of the damage's mechanical effects rather than the microstructure. Consider a simple one-dimensional case. The explanation of net area reduction leads to the concept of net stress $\bar{\sigma} = (\sigma/1-\omega)$ where ω is the damage variable and σ is the macroscopical stress. The introduction of the hypothesis of strain equivalence Lemaitre (1990) further leads to $E = E_0(1-\omega)$ where E and E_0 are the apparent and intrinsic Young's modulus, respectively. On the other hand, its equivalent form $\omega = 1-(E/E_0)$ demonstrates the true physical meaning of the damage variable. Ju (1989) developed the idea to the fullest by taking the elasticity tensor directly as the damage variable.

Some problems arise in extending the phenomenological damage definition to geomaterials: (1) Geomaterial stiffness is significantly dependent on the sign of stress/strain. Thus, the phenomenological damage variable also becomes stress/strain-dependent, which violates the independent principle of state variable. (2) The phenomenological damage measure cannot take into account the geological data of rock materials, which is measurable and provided important parameters for engineering designs.[1] (3) The phenomenological damage definition furnishes a simple damage-elasticity relation (e.g. $E = E_0(1-\omega)$) at the cost of damage evolution laws. Since the phenomenological damage definition lacks a definite microscopical meaning, the corresponding damage evolution law has been the weakest and most arbitrary aspect. (4) The triaxial generalization of the net stress leads to a nonsymmetric net stress rate tensor, which must further be symmetrized in some arbitrary ways in order to obtain a practical formulation, see e.g. Murakami (1988).

In general, these problems stems from the fact that the essence of damage is the distributed micro-defects, and its the mechanical behaviours presents in many aspects and generally there do not exist one-to-one correspondence between the behaviour and the damage. The relation between the phenomenological damage definition and the microstructures is not definite and unique but correlative, and the deviation sometimes becomes significant. In this paper, a second-order fabric tensor (see e.g. Oda-2, 1983 and Cowin, 1985), which is a geometric measure of material microstructure, is chosen as the damage variable. Consider a material sample with the size V^0 weakened by n microcracks. The damage tensor can be defined as

$$\mathbf{\Omega} = \frac{1}{V^0} \sum_{\alpha=1}^{n} r_\alpha^3 \mathbf{n}^\alpha \mathbf{n}^\alpha \tag{1}$$

where r_α and $\mathbf{n}^\alpha$ are the radius and normal vector of the α-th crack.

Based on such a damage tensor, the damage-dependent elasticity tensor cannot be deduced based on the hypothesis of strain. Some micromechanical attempts have been made to establish the relation, see e.g. Oda et al. (1984) and Lemaitre (1990). However, their deduced analytic expressions involve second- and fourth-rank fabric tensors and basically only furnish a linear

[1] Kawamoto et al. (1988) extended the damage theory to model jointed rock with the damage tensor based on statistical geological data. However, their efforts failed at the point where the geometrical damage tensor is not comparable with the classical damage theory.

approximation with respect to the principal values of the damage tensor. It will further complicate the expression to take into account crack interaction. Cowin (1985) developed the most general fabric tensor-dependent elasticity tensor by tensorial algebra, but its expression involves too many unknown parameters. In this paper, a one-parameter damage-dependent elasticity tensor is deduced by tensorial algebra and thermodynamic requirements, see eqns (15) and (20), which furnishes a fourth-order approximation.

In geomaterials, the damaging process is often coupled with plastic flow. Certain special loading purposes, e.g. hydraulic fracturing, have significant contribution to damage propagation. To describe such complicated mechanisms, only conjugate-force-based damage evolution laws can keep a unitary and compact form. As summarized by Chow and Lu (1989), many damage evolution laws of second damage tensors, e.g. from Chaboche, Lee, Murakami–Ohno, Sidoroff–Cordebois, etc., can be covered by linear irreversible thermodynamics, $\dot{\mathbf{\Omega}} = \mathbf{J} : \mathbf{Y}$ where $\mathbf{Y}$ is the thermodynamic force conjugate to the damage tensor and $\mathbf{J}$ is a fourth-order damage characteristic tensor. The characteristic tensor $\mathbf{J}$ has lacked of a definite definition due to the phenomenological attribute to the damage tensor. In Part II of this paper, an analytic expression of $\mathbf{J}$ is developed in Rice's (1971) normality structure following strictly the definition of the fabric tensor.

It is arguable to apply conjugate-force-based damage evolution laws to geomaterials. As discussed in Part II, these laws are suitable to a class of materials in which the influence of the macroscopic stress/strain on each defect appears only through the conjugate forces. In geomaterials, the macroscopic stress/strain changes the contact manner (close or open) of crack surfaces which in turn changes the crack propagation rule quantitatively and qualitatively, so it is difficult to exclude the macroscopic stress/strain explicitly from damage evolution laws. To overcome the difficulty and make a practical formulation, some authors simply assume that the driving force behind the damage propagation is the maximum tensile net stress. Obviously, it is a very coarse assumption. Damage generally propagates not along the direction of its conjugate force; the current existing damage also has influence on the direction (mainly through $\mathbf{J}$).[2] Furthermore, the conjugate force is much better than the stress to take into account other mechanisms contributing to damage propagation. In this paper, the conjugate-force-based damage evolution law is used and the influence of macroscopic stress/strain is reflected implicitly by introducing an equivalent state.

2. Development of an equivalent state

In this section, an isotropic damage model is formulated in strain space. The constitutive equations of a damage model mainly consists of two parts: damage elasticity and damage evolution law. The damage elasticity is the damage-dependent elasticity tensor $\mathbf{D}$ which generally takes the form $\mathbf{D} = \mathbf{D}(\mathbf{\Omega}, \boldsymbol{\varepsilon})$ due to the influence of strain $\boldsymbol{\varepsilon}$. In order to avoid a complicated formulation, the real state of a solid is mapped into an equivalent state on which the constitutive equations with a simple form like $\mathbf{D} = \mathbf{D}(\mathbf{\Omega})$ and $\dot{\mathbf{\Omega}} = \mathbf{J} : \mathbf{Y}$ hold true with respect to any strain state.

[2] The governing principle is that this direction will make the system dissipate maximum energy for a certain damage growth.

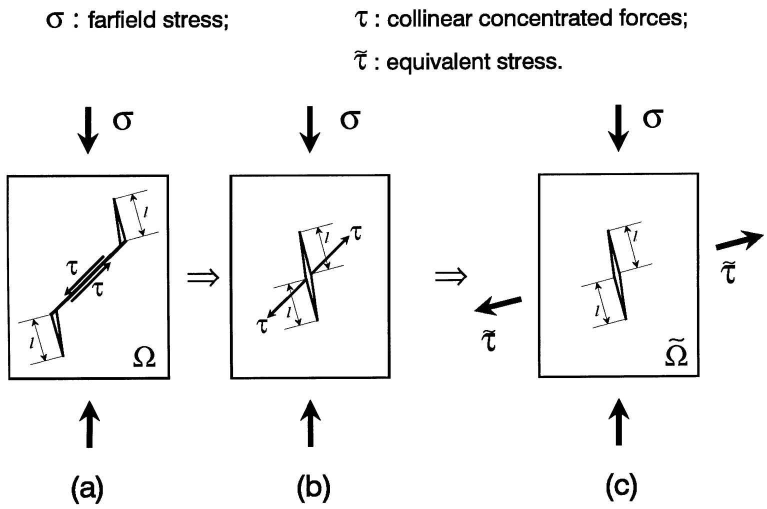


Fig. 1. Schematic frictional crack equivalence.

In geomaterials, the frictional sliding on crack surfaces plays a central role in either inelastic deformation or cracking. Sliding causes dilatation by opening the crack at asperities and by inducing local tensile cracking at some angle to the crack. Thus, the shear stress on the crack becomes the driving force of inelastic deformation and the materials become pressure-sensitive. In terms of macroscopic constitutive equations, sliding causes deviation from the normality structure, as pointed out by Rudnicki and Rice (1975).

Frictional crack models have been investigated by many authors, e.g. Nemat-Nasser and Obata (1988), Kachanov (1982), etc. For local tensile cracking, as shown in Fig. 1(a), the stress intensity factor of Mode I at crack tips can be computed with very good accuracy by considering an equivalent crack of length $2l$, subjected to a pair of collinear concentrated forces, τ, as well as the applied overall stress (see e.g. Nemat-Nasser and Obata, 1988), as shown in Fig. 1(b). In this representation, τ denotes the resultant force transmitted across the preexisting crack.

Macroscopically, the states (a) and (c) are characterized by two sets of state variables and functions, respectively, see Table 1. In this paper, the state (c) is termed the equivalent state; $\tilde{\mathbf{\Omega}}$ is termed the effective damage tensor, which corresponds to the equivalent cracks. Obviously, the fictitious cracks in the equivalent state are always in open state. The equivalent state is subjected to the effective stress $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \tilde{\boldsymbol{\tau}}$ where $\tilde{\boldsymbol{\tau}}$ is the macroscopic representation of τ, see Fig. 1(c). The plasticity effects are omitted in the formulation.

Let's consider the relations between the two states. It is postulated that the strain of the equivalent state under the effective stress is equal to the strain of the real state under the applied stress, namely $\boldsymbol{\varepsilon} = \tilde{\boldsymbol{\varepsilon}}$. For the equivalent state, the stiffness and damage variable possess the one-to-one correspondence, i.e. $\tilde{\mathbf{D}} = \mathbf{D}(\tilde{\mathbf{\Omega}})$ since $\tilde{\mathbf{\Omega}}$ corresponds to the open equivalent cracks. Therefore, the elasticity relation of the equivalent state, $\tilde{\boldsymbol{\sigma}} = \tilde{\mathbf{D}} : \tilde{\boldsymbol{\varepsilon}}$, leads to that of the real state

$$\boldsymbol{\sigma} = \mathbf{D}(\tilde{\mathbf{\Omega}}) : \boldsymbol{\varepsilon} - \tilde{\boldsymbol{\tau}} \tag{2}$$

The free energy function and the conjugate force of the equivalent state can be defined as

Table 1
Macroscopic characterization of real and equivalent states

	Real state (a)	Equivalent state (c)	Relations
Strain	$\boldsymbol{\varepsilon}$	$\tilde{\boldsymbol{\varepsilon}}$	$\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}$
Damage tensor	$\boldsymbol{\Omega}$	$\tilde{\boldsymbol{\Omega}}$	$\tilde{\boldsymbol{\Omega}} = \mathbf{P}^+ : \boldsymbol{\Omega}$
Stress	$\boldsymbol{\sigma}$	$\tilde{\boldsymbol{\sigma}}$	$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \tilde{\boldsymbol{\tau}}$
Elasticity tensor	$\mathbf{D}$	$\tilde{\mathbf{D}}$	$\tilde{\mathbf{D}} = \mathbf{D}(\tilde{\boldsymbol{\Omega}})$
Free energy	ϕ	$\tilde{\phi}$	$\phi = \tilde{\phi} = g\tilde{\boldsymbol{\Omega}} : \boldsymbol{\varepsilon}$
Conjugate force	$\mathbf{Y}$	$\tilde{\mathbf{Y}}$	$\mathbf{Y} = \mathbf{P}^+ :(\tilde{\mathbf{Y}} + g\boldsymbol{\varepsilon})$

$$\tilde{\phi} = \frac{1}{2}\boldsymbol{\varepsilon} : \tilde{\mathbf{D}} : \boldsymbol{\varepsilon}, \quad \tilde{\mathbf{Y}} = -\frac{\partial \tilde{\phi}}{\partial \tilde{\boldsymbol{\Omega}}} = \frac{1}{2}\boldsymbol{\varepsilon} : \frac{\partial \mathbf{D}(\tilde{\boldsymbol{\Omega}})}{\partial \tilde{\boldsymbol{\Omega}}} : \boldsymbol{\varepsilon} \tag{3}$$

Since τ acts on the equivalent crack as shown in Fig. 1(b), it is reasonable to assume that its equivalent stress $\tilde{\boldsymbol{\tau}}$ is parallel to $\tilde{\boldsymbol{\Omega}}$, i.e.

$$\tilde{\boldsymbol{\tau}} = g\tilde{\boldsymbol{\Omega}} \tag{4}$$

where g is assumed to be a non-negative material constant. Note that, in the real state, the relation $\boldsymbol{\sigma} = (\partial\phi/\partial\boldsymbol{\varepsilon})$ also leads to a stress–strain relation which should be consistent with eqn (2). Based on this point and eqn (4), the free energy function of the real state should be

$$\phi = \tfrac{1}{2}\boldsymbol{\varepsilon} : \mathbf{D}(\tilde{\boldsymbol{\Omega}}) : \boldsymbol{\varepsilon} - g\tilde{\boldsymbol{\Omega}} : \boldsymbol{\varepsilon} = \tilde{\phi} - g\tilde{\boldsymbol{\Omega}} : \boldsymbol{\varepsilon} \tag{5}$$

2.1. *Effective damage tensor*

In principle, the effective damage tensor $\tilde{\boldsymbol{\Omega}}$ should be computed directly using eqn (1) with respect to the equivalent cracks, but it is difficult to determine the equivalent cracks. Based on the fact that the equivalent cracks are roughly tensile-strain-oriented, the effective damage tensor is defined as

$$\tilde{\boldsymbol{\Omega}} = \mathbf{P}^+ : \boldsymbol{\Omega} \quad \text{where } P^+_{ijkl} = Q^+_{ik}Q^+_{jl}, \quad \mathbf{Q}^+ = \sum_{v=1}^{3} H(\varepsilon_v)\mathbf{p}^v\mathbf{p}^v \tag{6}$$

where $\mathbf{p}^v$ and ε_v are the v-th principal normal vector and value, respectively, of strain $\boldsymbol{\varepsilon}$. Evidently, the definition leads to $\tilde{\boldsymbol{\Omega}}_{ij} = \tilde{\boldsymbol{\Omega}}_{ji} \Rightarrow \tilde{\sigma}_{ij} = \tilde{\sigma}_{ji}$. The tensor $\mathbf{P}^+$ was termed positive projection tensor by Ju (1989) and Dragon et al. (1993).

Originally, $H(\cdot)$ was just the Heaviside function $\hat{H}(\cdot)$ and then the principal tensile strain can be determined as

$$\boldsymbol{\varepsilon}^+ = \sum_{v=1}^{3} \hat{H}(\varepsilon_v)\varepsilon_v\mathbf{p}^v\mathbf{p}^v = \mathbf{P}^+ : \boldsymbol{\varepsilon} \tag{7}$$

In some cases for geomaterials, the Heaviside function $\hat{H}$ will overestimate the difference between compression and tension. For example, in jointed rock, the joints are not perfectly contact but with some filling materials or roughness. Here H is defined as

$$H(\varepsilon_v) = (1-h)\hat{H}(\varepsilon_v)+h = \begin{cases} 1 & \text{if } \varepsilon_v > 0 \\ h & \text{otherwise} \end{cases} \tag{8}$$

where $h (0 \leqslant h < 1)$ is a material constant to reflect the properties of crack contact. In view of eqns (5) and (6), the conjugate force of the real state is

$$\mathbf{Y} = -\frac{\partial \phi}{\partial \mathbf{\Omega}} = \mathbf{P}^+ : (\tilde{\mathbf{Y}} + g\boldsymbol{\varepsilon}) \tag{9}$$

which is tensile-strain-oriented.

2.2. *Damage evolution law*

For the equivalent state, the conjugate-force-based damage evolution law holds true for any stress state

$$\mathrm{d}\tilde{\mathbf{\Omega}} = \mathrm{d}\tilde{\lambda}\tilde{\mathbf{J}} : \tilde{\mathbf{Y}} \tag{10}$$

where $\tilde{\mathbf{J}} = \mathbf{J}(\tilde{\mathbf{\Omega}})$. The function $\mathbf{J}(\mathbf{\Omega})$ is defined in eqns (93) and (84) of Part II of this paper:

$$\mathbf{J}(\mathbf{\Omega}) = \sum_{v=1}^{3} \omega_v^2 \left(4\mathbf{T}_v + \frac{9}{4}\mathbf{N}_v\right)$$

$$\mathbf{T}_v = T^v_{ijkl} = \tfrac{1}{4}(n^v_i n^v_k \delta_{jl} + n^v_i n^v_l \delta_{jk} + n^v_j n^v_k \delta_{il} + n^v_j n^v_l \delta_{ik}) - n^v_i n^v_j n^v_k n^v_l$$

$$\mathbf{N}_v = N^v_{ijkl} = n^v_i n^v_j n^v_k n^v_l \quad \text{(no summation for } v\text{)} \tag{11}$$

where $\mathbf{n}^v$ and ω_v ($v = 1, 2, 3$) are the principal directions and values, respectively, of damage tensor $\mathbf{\Omega}$. The associated damage surface is

$$\tilde{\mathscr{F}} = \tilde{\mathscr{G}} - \tilde{\mathscr{R}}, \quad \tilde{\mathscr{G}}^2 = \tfrac{1}{2}\tilde{\mathbf{Y}} : \tilde{\mathbf{J}} : \tilde{\mathbf{Y}} \quad \tilde{\mathscr{R}} = \max\{\mathscr{R}_0, \max \tilde{\mathscr{G}}\} \tag{12}$$

where $\mathscr{R}_0$ is the damage threshold of virgin materials. The damage multiplier is proposed as

$$\mathrm{d}\tilde{\lambda} = \beta\frac{\mathrm{d}\xi}{\tilde{\mathscr{G}}}, \quad \mathrm{d}\xi = |\mathrm{d}e_{ij}| = \sqrt{\mathrm{d}e_{ij}\,\mathrm{d}e_{ij}}, \quad e_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk} \tag{13}$$

where $\beta > 0$ is a material constant.

2.3. *Discussion on the equivalent state*

The postulate $\boldsymbol{\varepsilon} = \tilde{\boldsymbol{\varepsilon}}$ is just Lemaitre's (1990) hypothesis of strain equivalence that the strain associated with a damaged state under the applied stress is equivalent to the strain associated with its undamaged state under the effective stress if replacing the undamaged state with the equivalent state. One may argue that the undamaged state and its associated net stress can be used to furnish a much simpler formulation than can the equivalent state. Note that the net stress is not suitable to describe the anisotropic damaging, as discussed in the first section. It is also a very coarse

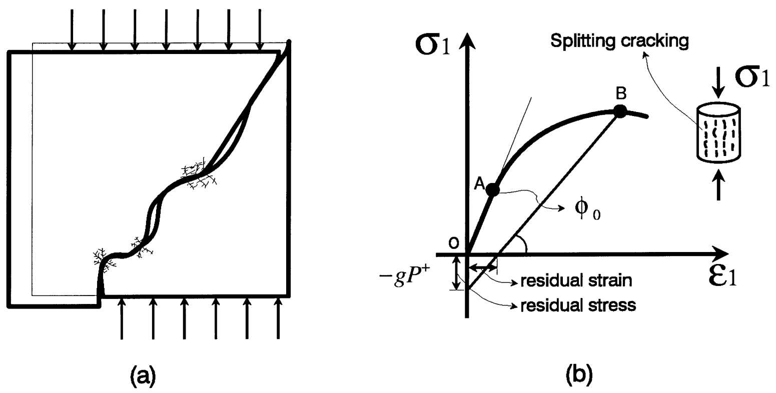


Fig. 2. (a) Schematic dilatancy mechanism; (b) typical stress–strain with residual stress and strain.

concept for anisotropic damage induced elasticity. For example, the simplest symmetric net stress may be defined as $(\bar{\sigma}_{ij}+\bar{\sigma}_{ji})/2$ which still leads to an unsymmetric elasticity tensor. One elasticity tensor is symmetric if $D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij}$.

The preceding formulation on the relations between the real state and the equivalent state is based on the three hypothesis: $\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}$, $\boldsymbol{\tau} = g\tilde{\boldsymbol{\Omega}}$, $\tilde{\boldsymbol{\Omega}} = \mathbf{P}^+ : \boldsymbol{\Omega}$ which all are approximate descriptions of their physical realities. From a purely mathematical viewpoint, the introduction of the two parameters h, g is used to adjust the basic relation $\boldsymbol{\sigma} = \mathbf{D}(\tilde{\boldsymbol{\Omega}}) : \boldsymbol{\varepsilon} - g\tilde{\boldsymbol{\Omega}}$ to fit the reality. From a different viewpoint, Dragon (1993) first introduced the term $g\boldsymbol{\Omega} : \boldsymbol{\varepsilon}^+ = g\tilde{\boldsymbol{\Omega}} : \boldsymbol{\varepsilon}$ in his free energy function to reflect $\boldsymbol{\Omega}$-generated residual stress/strain. This term is also important to describe dilatancy, see Fig. 2.

It should be pointed out that, the main purpose and advantage to develop the conjugate-force-based damage evolution law and the equivalent state, is not for a simple brittle solid but for complicated internal dissipative mechanisms and mechanical behaviour. The formulation can be easily extended to describe the process coupled with plasticity just be choosing an appropriate free energy function, e.g. $\tilde{\phi} = \frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p) : \tilde{\mathbf{D}} : \frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)$ where $\boldsymbol{\varepsilon}^p$ is the plastic strain. Then the influence of plasticity on damaging is reflected implicitly through the corresponding conjugate force $\tilde{\mathbf{Y}}$. The hydraulic fracturing mechanism can also be accommodated in the framework by this way, see Swoboda et al. (1995), Swoboda and Yang (1997) and Yang (1996). As usual, the damage's influence on plastic flow can be reflected implicitly through another effective stress base on another equivalent state (undamaged material). In this framework, such an effective stress $\tilde{\boldsymbol{\sigma}} = \mathbf{D}^0 : (\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)$ in line with Lemaitre's (1990) hypothesis of strain equivalence, where $\mathbf{D}^0$ is the isotropic elasticity tensor of the virgin material.

3. Damage elasticity

An explicit expression of the damage-dependent elasticity tensor $\mathbf{D}(\boldsymbol{\Omega})$ is developed in this section. In general, an elasticity tensor is subject to the general principles of continuum mechanics (see e.g. Malvern, 1969): (1) symmetry condition requires $D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij}$; (2) material symmetry condition requires that $\mathbf{D}(\boldsymbol{\Omega})$ be an isotropic tensor function; (3) positive definite condition requires the elastic potential function $W = \frac{1}{2}\varepsilon_{ij}D_{ijkl}\varepsilon_{kl}$ is positive-definite as a function of strain ε_{ij}. The isotropic tensor function $\mathbf{D}(\boldsymbol{\Omega})$, which satisfies the symmetric condition, takes the form

$$\begin{aligned} D_{ijkl} = {} & A_1\delta_{ij}\delta_{kl} + A_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ & + A_3(\Omega_{ij}\delta_{kl} + \Omega_{kl}\delta_{ij}) + A_4(\Omega_{ik}\delta_{jl} + \Omega_{il}\delta_{kl} + \Omega_{jk}\delta_{il} + \Omega_{jl}\delta_{ik}) \\ & + A_5\Omega_{ij}\Omega_{kl} + A_6(\Omega_{ik}\Omega_{jl} + \Omega_{il}\Omega_{jk}) \\ & + A_7(\Theta_{ij}\delta_{kl} + \Theta_{kl}\delta_{ij}) + A_8(\Theta_{ik}\delta_{jl} + \Theta_{il}\delta_{jk} + \Theta_{jk}\delta_{il} + \Theta_{jl}\delta_{ik}) \\ & + A_9(\Theta_{ij}\Omega_{kl} + \Omega_{ij}\Theta_{kl}) + A_{10}(\Theta_{ik}\Omega_{jl} + \Omega_{ik}\Theta_{jl} + \Theta_{il}\Omega_{jk} + \Omega_{il}\Theta_{jk}) \\ & + A_{11}\Theta_{ij}\Theta_{kl} + A_{12}(\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk}) \end{aligned} \tag{14}$$

where $\Theta_{ij} = \Omega_{im}\Omega_{mj}$; $A_1, A_2, \ldots, A_{12}$ are functions of invariants of Ω_{ij}. According to Cayley–Hamilton's theorem, the third- and higher-order terms can be expressed as linear combinations of linear and quadratic terms. Thus, terms like $\Omega_{im}\Theta_{mj}$, $\Theta_{im}\Theta_{mj}$ disappear in eqn (14), see e.g. Bazant (1983). Obviously, Murakami (1983), Stumvoll and Swoboda (1993), and Cowin's (1985) expressions of the damage elasticity are all special cases of eqn (14).

It is a very complicated problem to impose the positive definite condition on eqn (14). It is natural to consider such a strong condition to require all terms to be non-negative in any case. Obviously, the strong condition leads to $A_3 = A_4 = A_7 = A_0 = \cdots = 0$, which causes significant loss of generality. Here, the crux is solved by introducing an intermediate second-order tensor Φ_{ij} to define the elasticity tensor

$$D_{ijkl} = \lambda\Phi_{ij}\Phi_{kl} + \mu(\Phi_{ik}\Phi_{jl} + \Phi_{il}\Phi_{jk}), \quad \Phi_{ij} = C_1\delta_{ij} + C_2\Omega_{ij} + C_3\Omega_{im}\Omega_{mj} \tag{15}$$

where C_1, C_2 and C_3 are functions of the invariants of Ω_{ij}; λ and μ are two non-negative constants. Its expansion is

$$\begin{aligned} D_{ijkl} = {} & C_1C_1[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \\ & + C_1C_2[\lambda\Omega_{ij}\delta_{kl} + \lambda\Omega_{kl}\delta_{ij} + \mu(\Omega_{ik}\delta_{jl} + \Omega_{il}\delta_{jk} + \Omega_{jk}\delta_{il} + \Omega_{jl}\delta_{ik})] \\ & + C_2C_2[\lambda\Omega_{ij}\Omega_{kl} + \mu(\Omega_{ik}\Omega_{jl} + \Omega_{il}\Omega_{jk})] \\ & + C_1C_3[\lambda\Theta_{ij}\delta_{kl} + \lambda\Theta_{kl}\delta_{ij} + \mu(\Theta_{ik}\delta_{jl} + \Theta_{il}\delta_{jk} + \Theta_{jk}\delta_{il} + \Theta_{jl}\delta_{ik})] \\ & + C_2C_3[\lambda\Theta_{ij}\Omega_{kl} + \lambda\Omega_{ij}\Theta_{kl} + \mu(\Theta_{ik}\Omega_{jl} + \Omega_{ik}\Theta_{jl} + \Theta_{il}\Omega_{jk} + \Omega_{il}\Theta_{jk})] \\ & + C_3C_3[\lambda\Theta_{ij}\Theta_{kl} + \mu(\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk})] \end{aligned} \tag{16}$$

which is a complete tensorial polynomial as compared with eqn (14). Then the positive definite condition requires

$$W = \frac{1}{2}\varepsilon_{ij}D_{ijkl}\varepsilon_{kl} = \frac{\lambda}{2}(\Phi_{ij}\varepsilon_{ij})^2 + \mu\varepsilon_{im}\Phi_{mj}\Phi_{in}\varepsilon_{nj} \geqslant 0 \tag{17}$$

Note that $\varepsilon_{im}\Phi_{mj}\Phi_{in}\varepsilon_{nj} \geqslant 0$ if Φ_{ij} is a positive definite tensor. Therefore, a fourth-order positive definite problem reduced to a second-order one.

Although not necessarily, the normalized damage variable is convenient to use. Here it is postulated that

$$\mathbf{D} = \begin{cases} \mathbf{D}^0 & \text{if}\, \boldsymbol{\Omega} = \mathbf{0} \\ \mathbf{0} & \text{if}\, \boldsymbol{\Omega} = \mathbf{I} \end{cases} \tag{18}$$

where $\mathbf{I} = \delta_{ij}$ and $\mathbf{D}^0$ is the isotropic elasticity tensor of the virgin material. Based on this condition, λ and μ in eqn (15) are just Lamé's constants of virgin material and C_1, C_2, C_3 should satisfy

$$C_1 = 1 \quad \text{if}\, \boldsymbol{\Omega} = \mathbf{0}; \quad C_1 + C_2 + C_3 = 0 \quad \text{if}\, \boldsymbol{\Omega} = \mathbf{I} \tag{19}$$

If further assuming that C_1, C_2, C_3 are material constants and noting the positive definite requirement on Φ_{ij}, we have

$$C_1 = 1, \quad C_2 = -k, \quad C_3 = -(1-k), \quad 0 \leqslant k \leqslant 1 \tag{20}$$

so there is only one independent material constant k. The elasticity tensor defined in eqns (15) and (20) represents the orthotropic symmetry of elasticity. The so-called "hypothesis of complementary energy equivalence", see e.g. Lee et al. (1985) and Zhang (1992), leads to $\mathbf{D} = (\mathbf{I}-\boldsymbol{\Omega})\cdot\mathbf{D}^0\cdot(\mathbf{I}-\boldsymbol{\Omega})$, which is just a special case of eqn (15) for $k = 1$.

4. Numerical results

Several numerical examples from simple specimens to practical structural analyses, have been worked out to illustrate the behaviour of the damage model.

4.1. *Damage-dependent elasticity tensor*

The behaviour of the damage-dependent elasticity tensor $\mathbf{D}(\tilde{\boldsymbol{\Omega}})$ defined in eqn (15) and (6) is illustrated by one set of uniaxial tests, as shown in Fig. 3. The parallel microcracks in the specimen

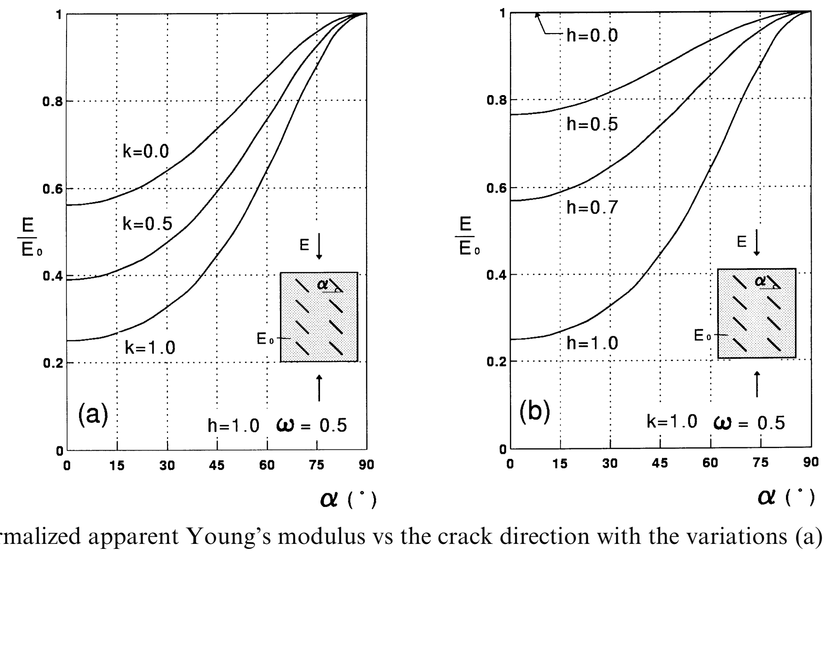


Fig. 3. Normalized apparent Young's modulus vs the crack direction with the variations (a) k and (b) h.

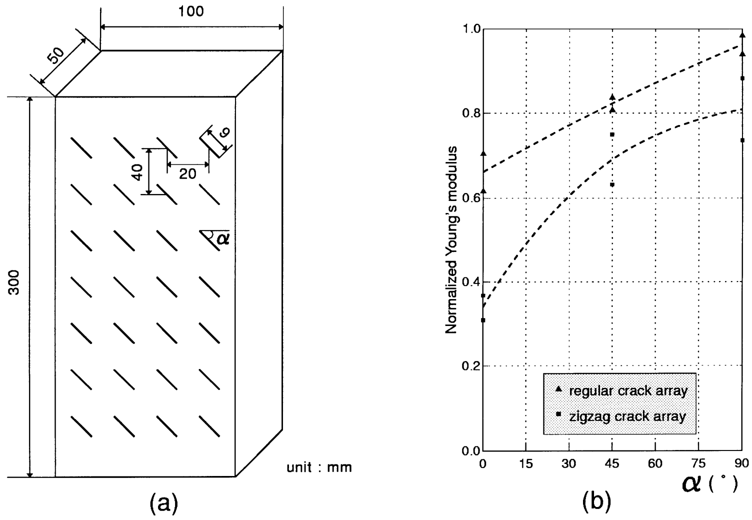


Fig. 4. (a) Experimental specimen; (b) apparent Young's modulus with changing crack direction.

can be characterized by a damage vector (n_i, ω), where $\omega = 0.5$ and the normal vector n_i is determined by α, the angle between loading direction and the damage vector. The damage vector corresponds to a damage tensor $\Omega_{ij} = \omega n_i n_j$. Then, eqn (15) is graphed into curves illustrating the relation between the apparent Young's modulus along the loading direction and α with different k and h. In test (a), we change k but keep $h = 1.0$ which implies that $\tilde{\Omega}_{ij} = \Omega_{ij}$. In test (b), we change h but keep $k = 1.0$, and the results indicate that the Heaviside function ($h = 0.0$) overestimates the compressive effects for geomaterials. The relation between apparent Young's modulus along the loading direction and α is also confirmed by the experiment done by Kawamoto et al. (1988), as shown in Fig. 4.

4.2. *Uniaxial and biaxial loading tests*

A uniaxial loading process is illustrated in Fig. 5. The material is initially virgin and the parameters are listed in Table 2. As a proportional loading process, $\dot{\mathbf{\Omega}}$, $\mathbf{\Omega}$, $\mathbf{Y}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ are all coaxial. Their process curves vs the axial strain are depicted in Fig. 5. The damage propagates mainly along the lateral direction in which the tensile strain is developed, as suggested by the conjugate force components Y_{11} and Y_{22}. The test reflects the typical "splitting" mechanism in geomaterials.

The biaxial loading test with the same parameters is shown in Fig. 6. It indicates that the model can describe the main deformation characters of geomaterials with few parameters. Using a conventional orthotropic elasticity model it is difficult to reflect such significant dilatancy.

4.3. *Parameter identification*

Quite obviously, the parameter identification can only be a trial-and-error procedure. By fitting the experimental curve of quartzite performed by Bieniawski (1968), one set of parameters is

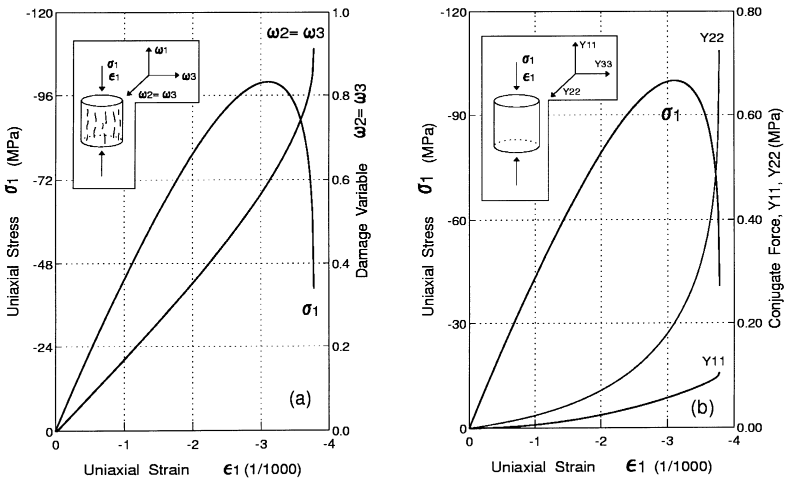


Fig. 5. (a) Axial stress and damage variable; (b) axial stress and conjugate forces under increasing axial strain.

Table 2
Material parameters for uniaxial and biaxial loading tests

E	50,000.0	MPa
v	0.25	—
h	0.1	—
k	1.0	—
$\mathscr{G}_0$	0.3	MPa
β	700,000.0	—
g	50.0	MPa

obtained, as shown in Fig. 7. In order to identify the parameters efficiently, some parameter sensitivity tests have been done by Yang (1996).

4.4. *Structural analysis cases*

A 2-D structural analyses case concerns the Kölnbrein arch dam in Austria. There were cracking accidents near the dam heel, as shown in Fig. 8(a), during the filling of the reservoir in 1978 and 1983, see e.g. Linsbauer et al. (1989). The figures (b) and (c) are the calculation area and mesh, respectively, with 913 elements and 2700 nodes. The figures from (d)–(f) show the process of damage propagation during the loading. The figure (g) is the same state as (f) with the vector representation of the damage tensor. The tendency of the calculation is roughly consistent with Linsbauer's (1989) result of linear elastic fracture mechanics.

A 3-D structural analysis has been performed on the Xiaowan arch dam which is under design and located in the upstream of Mekong River, China, see Fig. 9. It will be the highest arch dam in

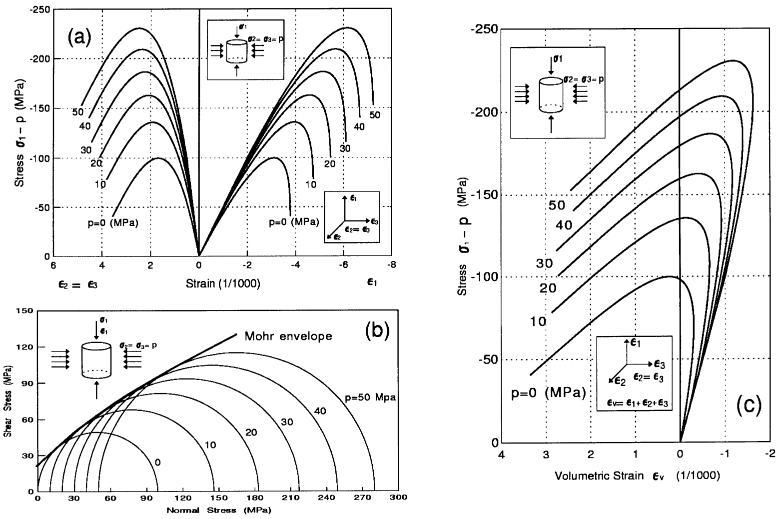


Fig. 6. Biaxial compressive tests of proposed damage model; (a) axial and lateral strain under increasing axial stress; (b) Mohr envelope for strength failure; (c) relationship between volumetric strain and axial stress.

E	84217.0	MPa
ν	0.1875	–
h	0.1	–
k	1.0	–
$\mathcal{G}_0$	77.0	MPa
β	41600.0	–
g	65.0	MPa

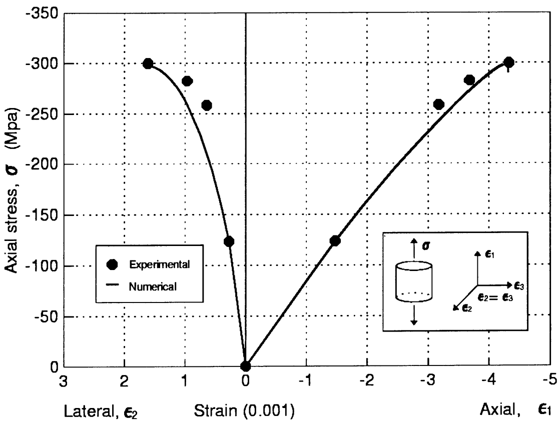


Fig. 7. Parameter identification for quartzite.

the world with the height of 292 metres and a planned installation 4200 MW. The FEM mesh includes 678 elements and 3366 nodes, as shown in Fig. 10(a). It is illustrated that the damage is mainly concentrated on the damage heel of the crown cantilever, as shown in Fig. 10(b). The same structural problem has also been analyzed by Yang (1996) with the contact elements developed by Swoboda and Lei (1994) but it is difficult to predict the real cracking trajectory due to their fixed location. Based on the two calculations, it is suggested that some appropriate preventive measures against cracking should be made to the dam.

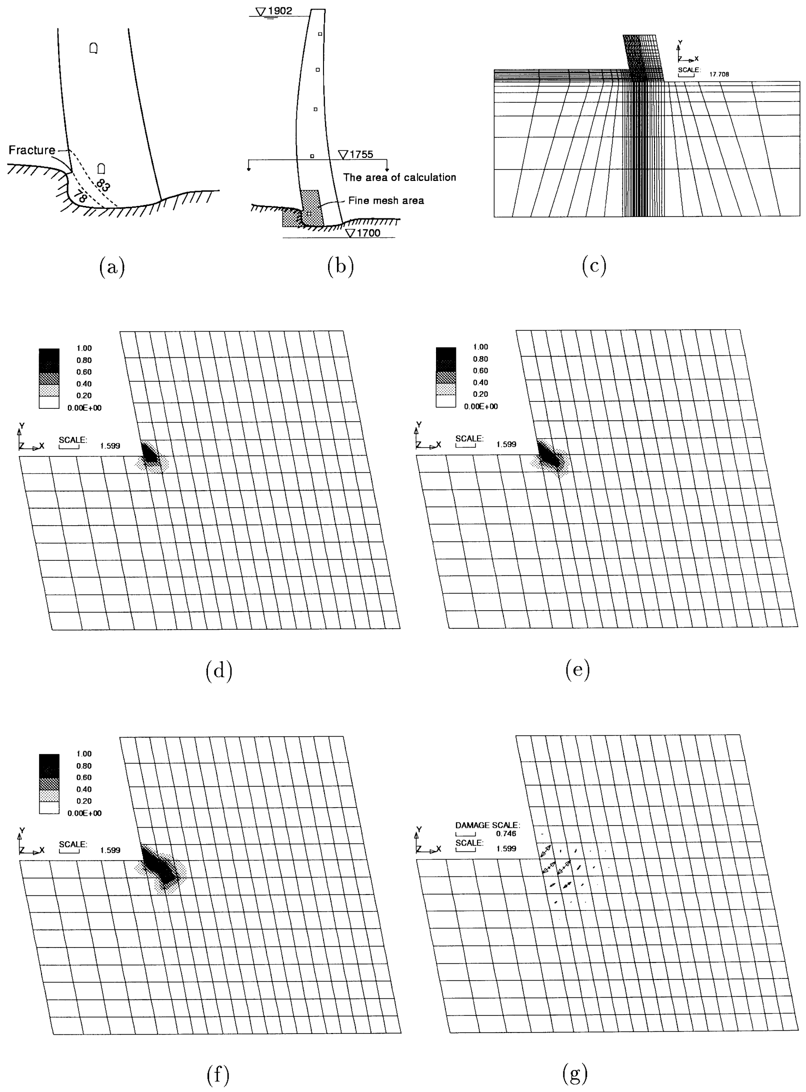
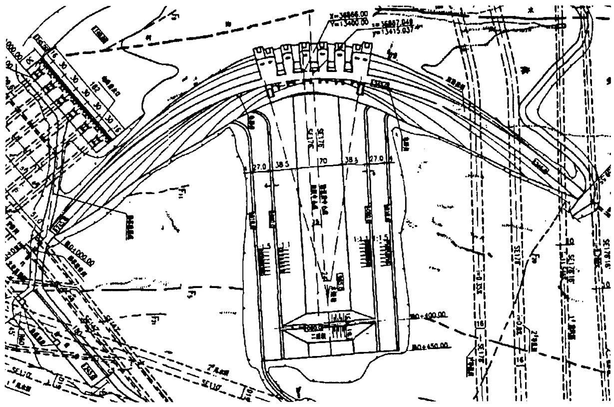


Fig. 8. Kölnbrein Arch Dam analysis: (a) cracking area; (b) calculation area; (c) calculation mesh; (d), (e), (f) distribution of the maximum principal damage value in the fine mesh area for differfent loading steps; (g) vector representation of the damage tensor distribution at the last loading step.

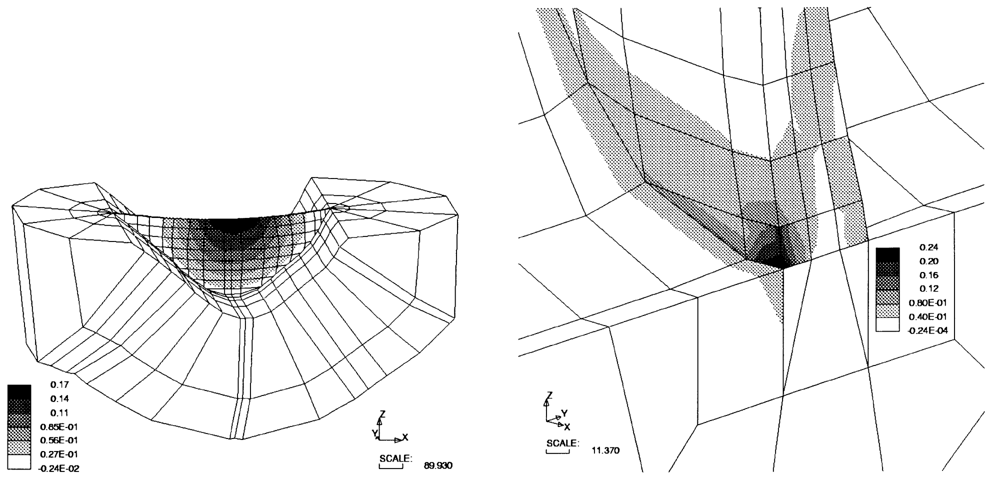
5. Conclusion

A microstructural damage variable, e.g. the second-order damage tensor employed in this paper, furnishes a much more essential characterization of distributed micro-defects than does a phenomenological damage variable, especially in geomaterials. Starting from it, a one-parameter damage-dependent elasticity tensor is developed based on tensorial algebra and thermodynamics requirements; a conjugate-force-based anisotropic damage evolution law is deduced within the



E	21000	MPa
ν	0.167	–
h	0.01	–
k	0.6	–
$\mathcal{G}_0$	15.1	MPa
β	16500.0	–
g	85.0	MPa

Fig. 9. Layout of Xiaowan Hydropower Station and concrete parameters.



(a) (b)

Fig. 10. (a) Displacement along river; (b) maximum damage distribution around the dam heel at the loading step: dead load + water load.

framework of Rice's (1971) "normality structure" (in Part II). Although the deduction is complicated and not straightforward as compared with the deduction based on the concept of net stress, it turns out a practical formulation with a rigorous basis.

A basic assumption behind the deduction in the constitutive laws of the micro-defects hold true for any macroscopic stress state. Evidently, the deduced constitutive equations based on the assumption represent certain ideal materials. In order to extend the deduced constitutive equation to geomaterials, the real state of geomaterials is mapped into an equivalent state of the ideal materials. The equivalent state represents current open cracks in terms of microstructure and is subjected to the effective stress which can be used to simulate "splitting cracking", dilatancy and residual strain/stress. The effective stress and equivalent state have a parallel relation with the net stress and the associated undamaged state. Of the two, however, the latter can only furnish a very coarse description of anisotropic damage, as indicated in the paper.

Acknowledgements

The work reported here was performed under contract No. S08004-TEC for the Sino-Austrian cooperative project "Damage Tensor", which was supported by the Austrian National Science Foundation (Fonds zur Förderung der Wissenschaften).

References

Bazant, Z.P., 1983. Comment on orthotropic models for concrete and geomaterials. J. Eng. Mech. ASCE 109, 849–865.
Bieniawski, Z.T., 1968. Fracture dynamics of rock. The International Journal of Fracture Mechanics 4, 415–430.
Chow, C.L., Lu, T.J., 1989. On evolution laws of anisotropic damage. Eng. Fracture Mech. 34, 679–701.
Cowin, S.C., 1985. The relationship between the elasticity tensor and the fabric tensor. Mech. of Mat. 4, 137–147.
Dragon, A., Mroz, Z., 1979. A continuum model for plastic-brittle behaviour of rock and concrete. Int. J. Engng. Sci. 17, 121–137.
Dragon, A., Charlez, Ph., Pham, D., Shao, J.F., 1993. A model of anisotropic damage by (micro)crack growth. In Proceedings of the International Symposium on Assessment and Prevention of Failure Phenomena in Rock Engineering, ed. A. G. Pasamehmetoglu, et al., pp. 71–78. Istanbul, Turkey.
Ju, J.W., 1989. On energy-based coupled elastoplastic damage theories: constitutive modelling and computational aspects. International Journal of Solids and Structures 25, 803–833.
Ju, J.W., Lee, X., 1991. Micromechanical damage models for brittle solids. J. Eng. Mech. ASCE 117, 1495–1536.
Kachanov, L.M., 1958. Time of the Rupture process under creep conditions. IVZ Akad Nauk, S.S.R., Old Tech Nauk 8, 26–31.
Kachanov, M., 1982. A microcrack model of rock inelasticity part II: propagation of microcracks. Mech. Mater. 1, 29–41.
Kawamoto, T., Ichikawa, Y., Kyoya, T., 1988. Deformation and fracturing behaviour of discontinuous rock mass and damage mechanics theory. Int. J. Numer. Anal. Meth. Geomech. 12, 1–30.
Lee, H., Peng, K., Wang, J., 1985. An anisotropic damage criterion for deformation instability and its application to forming limit analysis of metal plates. Eng. Fracture Mech. 21, 1031–1054.
Lemaitre, J., 1990. A Course on Damage Mechanics, Second Edition, Springer.
Linsbauer, H. N., Ingraffea, A. R., Rossmanith, H. P. and Wawrzynek, P. A. (1989). Simulation of cracking in large arch dam. J. Struct. Eng. ASCE 115, 1599–1630.
Malvern, L.E., 1969. Introduction of the Mechanics of a Continuous Medium. Prentice-Hall, Inc., Englewood Cliff, N.J.
Murakami, S., 1983. Notion of continuum damage mechanics and its application to anisotropic creep damage theory. J. Eng. Mater. Technol. 105, 99–105.
Murakami, S., 1988. Mechanical modeling of material damage. Journal of Applied Mechanics 55, 280–286.
Nemat-Nasser, S. and Obata, M. (1988) A microcrack model of dilatancy in brittle materials. J. Eng. Mech. ASCE 55, 24–35.
Oda, M., 1983. A method for evaluating the effect of crack geometry on the mechanical behaviour of cracked rock masses. Mech. Mater. 2, 163–171.
Oda, M., Suzuki, K., Maeshibu, T., 1984. Elastic compliance for rock-like materials with random cracks. Solids and Foundations 24, 27–40.
Rice, J.R., 1971. Inelastic constitutive relations for solids: an integral variable theory and its application to metal plasticity. Journal of the Mechanics and Physics of Solids 19, 433–455.

Rudnicki, J.W., Rice, J.R., 1975. Conditions for the localization of deformation in pressure-sensitive dilatancy materials. Journal of the Mechanics and Physics of Solids 23, 371–394.

Stumvoll, M., Swoboda, G., 1993. Deformation behavior of ductile solids containing anisotropic damage. J. Eng. Mech. ASCE 119, 1331–1352.

Swoboda, G., Lei, X.Y., 1994. Simulation of arch dam—foundation interaction with a new friction interface element. Int. J. Numer. Anal. Meth. Geomech. 17, 601–617.

Swoboda, G., Yang, Q., Ito, F., 1995. Damage propagation model and its application to rock engineering problems. In Proceedings of the Eighth International Congress on Rock Mechanics, ed. T. Fujii, pp. 159–164. Tokyo, Japan.

Swoboda, G., Yang, Q., 1997. On energy-based anisotropic damage models. In Proceedings of the 9th IACMAG Conference. Wuhan, China.

Yang, Q., 1996. Numerical modelling of discontinuous geomaterials considering damage propagation and seepage. Ph.D. dissertation, Faculty of Architecture and Civil Engineering, University of Innsbruck, Austria.

Zhang, W.H., 1992. Numerical analysis of continuum damage mechanics. Ph.D. dissertation, School of Civil Engineering, University of New South Wales, Australia.